Domaine Decomposition and optimal control for the inversion of pressure in volcanic fractures

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Linear elastostatic crack problem

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H}^1(\Omega) \text{ such that :} & \\ -\mathrm{div}\,\sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{in } \Gamma_D, \\ \sigma(\mathbf{u})\cdot\mathbf{n} = 0 & \text{on } \Gamma_N, \\ \sigma(\mathbf{u})\cdot\mathbf{n} = \mathbf{p_n} & \text{on } \Gamma_C. \end{cases}$$

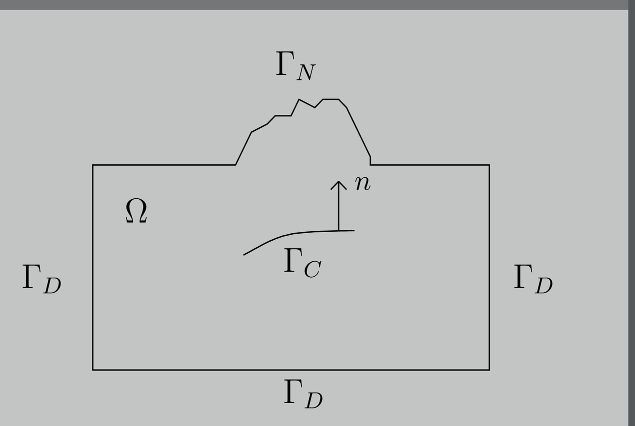

Figure 1:Volcanic cracked domain

Domain decomposition

$$\begin{cases} \text{Find } \mathbf{u}^{\pm} \in \mathbf{H}^1(\Omega) \text{ such that :} & \\ -\mathrm{div}\,\sigma(\mathbf{u}^{\pm}) = \mathbf{f}^{\pm} & \text{in } \Omega^{\pm}, \\ \mathbf{u}^{\pm} = 0 & \text{on } \Gamma_D^{\pm}, \\ (\sigma(\mathbf{u})\cdot\mathbf{n})^{\pm} = 0 & \text{on } \Gamma_N^{\pm}, \\ (\sigma(\mathbf{u})\cdot\mathbf{n})^{\pm} = \mathbf{p}\cdot\mathbf{n}^{\pm} & \text{on } \Gamma_C, \\ [[\mathbf{u}]] = 0 & \text{on } \Gamma_0, \\ [[\sigma(\mathbf{u})]]\cdot\mathbf{n}^{+} = 0 & \text{on } \Gamma_0, \end{cases}$$

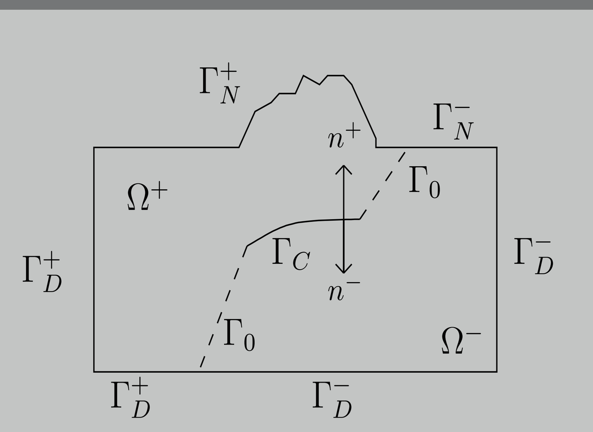

Figure 2:Splitting the domain

Variational formulation

Lagrangian of the direct problem

$$\mathcal{L}(\mathbf{u}^+,\mathbf{u}^-,\lambda) = \frac{1}{2}\mathbf{a}^+(\mathbf{u}^+,\mathbf{u}^+) + \frac{1}{2}\mathbf{a}^-(\mathbf{u}^-,\mathbf{u}^-) - \mathbf{l}^+(\mathbf{u}^+) - \mathbf{l}^-(\mathbf{u}^-) + \mathbf{b}(\lambda,\mathbf{u}^+) - \mathbf{b}(\lambda,\mathbf{u}^-)$$

- $\mathbf{a}^{\pm}(\mathbf{u}^{\pm},\mathbf{v}^{\pm}) = \int_{\Omega^{\pm}} \sigma(\mathbf{u}^{\pm}) : \varepsilon(\mathbf{v}^{\pm})\,d\Omega^{\pm}$
- $\mathbf{l}^{\pm}(\mathbf{v}^{\pm}) = \int_{\Omega^{\pm}} \mathbf{f}\cdot\mathbf{v}^{\pm}\,d\Omega^{\pm} + \int_{\Gamma_C} (\mathbf{pn})^{\pm}\cdot\mathbf{v}^{\pm}\,d\Gamma_C$
- $\mathbf{b}(\lambda,\mathbf{v}) = (\lambda,\mathbf{v})_{\Gamma_0} = \int_{\Gamma_0} \lambda\cdot\mathbf{v}\,d\Gamma_0 \longrightarrow$ constraints on Γ_0

Saddle point conditions

Find $(\mathbf{u}^+,\mathbf{u}^-,\lambda) \in \mathbf{V}^+ \times \mathbf{V}^- \times \mathbf{W}$ such that

$$\mathbf{a}^+(\mathbf{u}^+,\mathbf{v}^+) + \mathbf{b}(\lambda,\mathbf{v}^+) = \mathbf{l}^+(\mathbf{v}^+), \quad \forall \mathbf{v}^+ \in \mathbf{V}^+,$$
$$\mathbf{a}^-(\mathbf{u}^-,\mathbf{v}^-) - \mathbf{b}(\lambda,\mathbf{v}^-) = \mathbf{l}^-(\mathbf{v}^-), \quad \forall \mathbf{v}^- \in \mathbf{V}^-,$$
$$\mathbf{b}(\mu,\mathbf{u}^+) - \mathbf{b}(\mu,\mathbf{u}^-) = 0, \quad \forall \mu \in \mathbf{W}.$$

where $\mathbf{V}^+$, $\mathbf{V}^-$ and $\mathbf{W}$ are natural function spaces for the problem.

Optimal control (inversion) problem

- $\mathbf{u_d}$: observation data
- $\mathbf{C}$: observation error covariance
- $\alpha > 0$: regularization parameter

Cost function

$$\mathbf{J}(\mathbf{p}) := \frac{1}{2}\int_{\Gamma_N^{\pm}} (\mathbf{u}^{\pm} - \mathbf{u_d})^T \mathrm{C}^{-1} (\mathbf{u}^{\pm} - \mathbf{u_d})\,d\Gamma_N^{\pm} + \frac{\alpha}{2}\|\mathbf{p}\|^2_{L^2(\Gamma_C)}.$$

Goal : find $\mathbf{p}$ minimizing $\mathbf{J}$ i.e. such that the computed field $\mathbf{u}$ fits best the observed data.

Sensitivity analysis - Optimality conditions

Gradient of J

$$\nabla \mathbf{J}(\mathbf{p}) = \mathbf{z}\cdot\mathbf{n} + \alpha\mathbf{p}.$$

where $\mathbf{z}$ solves the adjoint state equation

$$\mathbf{a}(\mathbf{z},\mathbf{v}) = \mathbf{c_N}(\mathbf{u} - \mathbf{u_d},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}.$$

Optimal descent step length at point p in the direction d

$$\rho^* = -\left[\mathbf{c_N}(\mathbf{u} - \mathbf{u_d},\delta\mathbf{u}) + \alpha(\mathbf{p},\mathbf{d})_{\Gamma_C}\right]\left[\mathbf{c_N}(\delta\mathbf{u},\delta\mathbf{u}) + \alpha(\mathbf{d},\mathbf{d})\right]^{-1}.$$

Notations

- $\mathbf{c_N}(\mathbf{u},\mathbf{v}) = \int_{\Gamma_N^{\pm}} (\mathbf{u}^{\pm})^T \mathrm{C}^{-1} \mathbf{v}^{\pm}\,d\Gamma_N^{\pm}$
- the displacement $\delta\mathbf{u}$ is the state obtained with pressure $\mathbf{d}$ on the crack

Optimization algorithm - Conjugate gradient

- Initialization : $\mathbf{p_0}$ is given
  - Compute the initial state $(\mathbf{u}_0^{\pm},\lambda_0^{e})$, adjoint state $(\mathbf{z}_0^{\pm},\lambda_0^{a})$.
  - Initial gradient : $\mathbf{g_0} = \mathbf{z}_0^{\pm}\cdot\mathbf{n}^{\pm} + \alpha\mathbf{p_0}$. Initial direction : $\mathbf{d_0} = -\mathbf{g_0}$
- Iteration $\mathbf{k} \geq 0$
  - Sensitivity: Compute $(\mathbf{w_k},\lambda_k^{s})$ such that
    $$\mathbf{a}^{\pm}(\mathbf{w}_k^{\pm},\mathbf{v}^{\pm}) + \mathbf{b}(\lambda_k^{s},\mathbf{v}^{\pm}) = (\mathbf{d}\mathbf{n}^{\pm},\mathbf{v}^{\pm})_{\Gamma_C}, \quad \forall \mathbf{v}^{\pm} \in \mathbf{V}^{\pm},$$
    $$\mathbf{b}(\mu,\mathbf{w}_k^{+}) - \mathbf{b}(\mu,\mathbf{w}_k^{-}) = 0, \quad \forall \mu \in \mathbf{W}.$$
  - Compute the stepsize ρ_k
  - Update $\mathbf{p_{k+1}} = \mathbf{p_k} + \rho_k\mathbf{d_k}$, $\mathbf{u}_{k+1}^{\pm} = \mathbf{u}_k^{\pm} + \rho_k\mathbf{w}_k^{\pm}$
  - Adjoint Compute $(\mathbf{z}_k^{\pm},\lambda_k^{a})$ such that
    $$\mathbf{a}^{\pm}(\mathbf{z}_k^{\pm},\mathbf{v}^{\pm}) + \mathbf{b}(\lambda_k^{a},\mathbf{v}^{\pm}) = \mathbf{c_N}(\mathbf{u}_k^{\pm} - \mathbf{u}_d^{\pm},\mathbf{v}^{\pm}), \quad \forall \mathbf{v}^{\pm} \in \mathbf{V}^{\pm},$$
    $$\mathbf{b}(\mu,\mathbf{z}_k^{+}) - \mathbf{b}(\mu,\mathbf{z}_k^{-}) = 0, \quad \forall \mu \in \mathbf{W}.$$
  - New gradient: $\mathbf{g_{k+1}} = \mathbf{z}_{k+1}^{\pm}\cdot\mathbf{n}^{\pm} + \alpha\mathbf{p_{k+1}}$
  - New direction: $\mathbf{d_{k+1}} = -\mathbf{g_{k+1}} + \beta_k\mathbf{d_k}$, $\beta_k = (\mathbf{g_{k+1}},\mathbf{g_{k+1}})_{\Gamma_C}(\mathbf{g_k},\mathbf{g_k})_{\Gamma_C}^{-1}$

Numerical simulations - academic 3D test

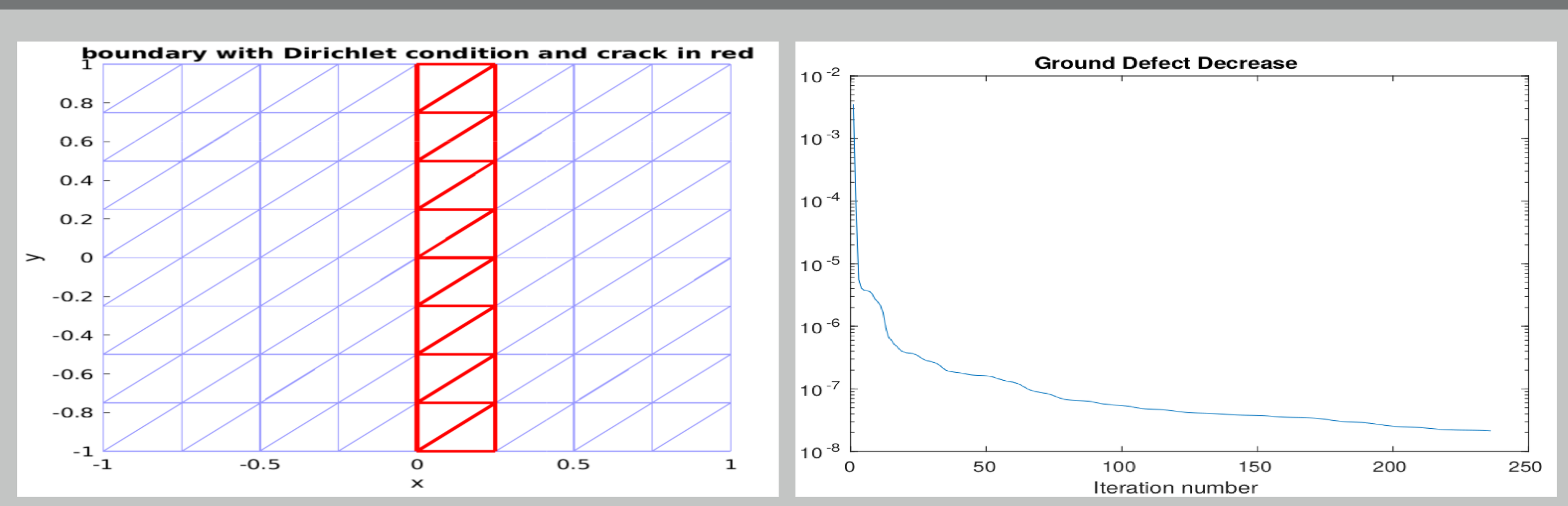

Figure 3:Improvement the convergence of the approximated solution to analytical solution with local refinement and optimal control.

- discretization by the finite element method using the GETFEM++ finite element (free) library
- Programs written in Python or Matlab.

A realistic case - Piton de la fournaise

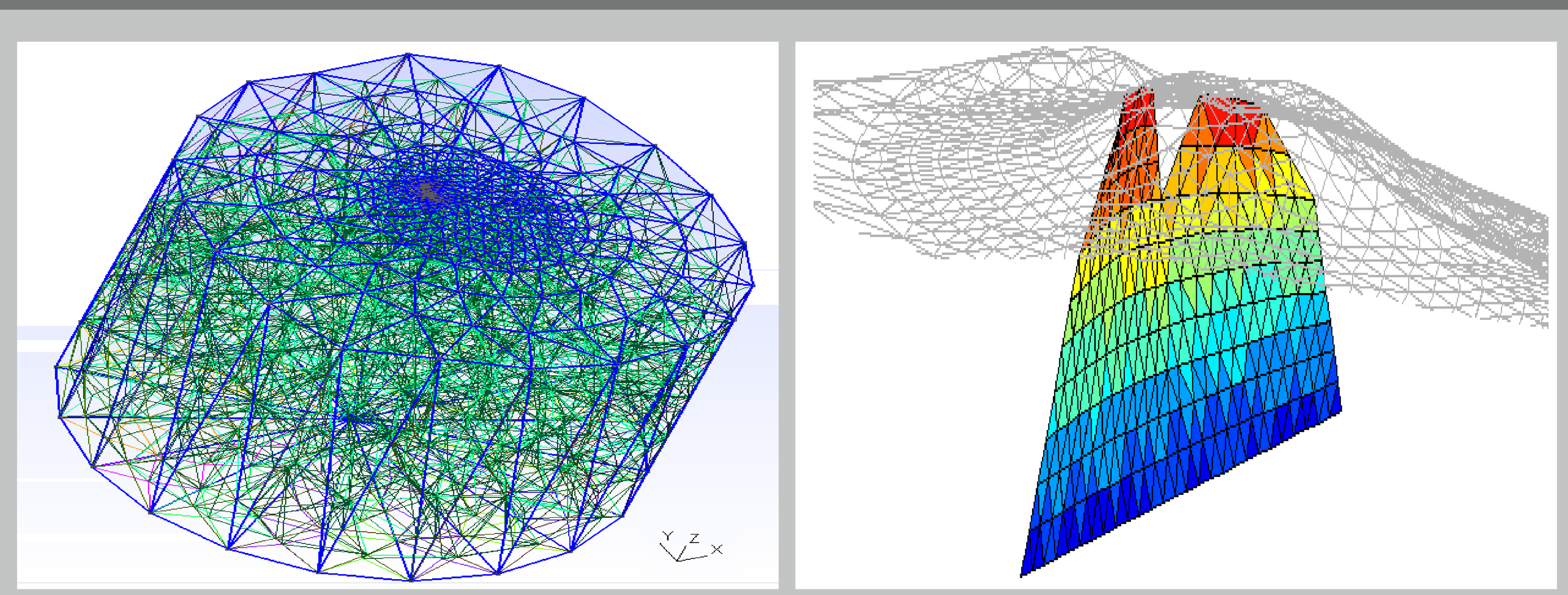

Figure 4:Mesh generated from topographic data provided by IGN, with digital elevation model (DEM) and Gmsh (left). Dike triangular surface mesh, representing the crack (right)

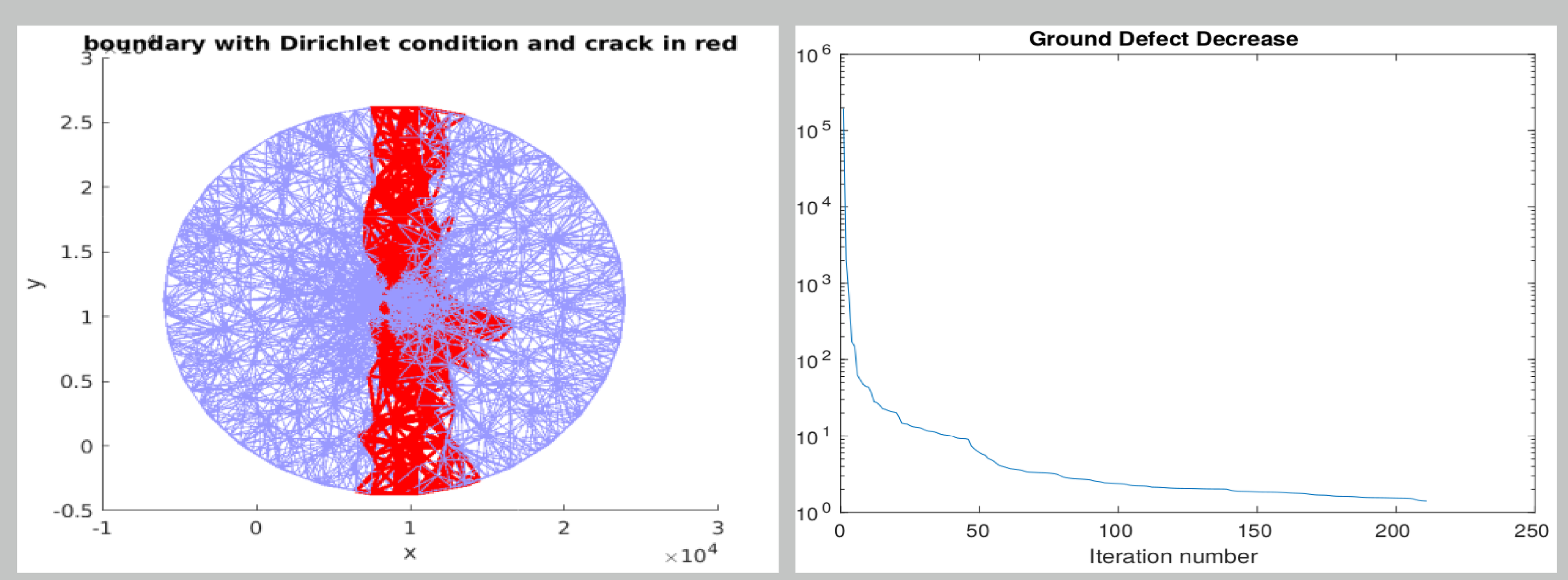

Figure 5:Elements of the crack (red) and Ground error decrease

Perspectives

- Implement preconditionners to improve the convergence in 3D realistic computations
- Fracture identification (localization and geometries) by the topological gradient method and shape optimization